Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 8 Parametric surface representation

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = r_u \times r_v$ of the surface.

1. *xy*-plane **r**(u, v) = (u, v) (thus u **i** + v **j**; similarly in problems 2 - 8).

```
Clear["Global`*"]
```

I just looked at the site *https://math.stackexchange.com/questions/152467/parametric-form-of-a-plane*, where it tells how to express a plane parametrically. There it says the equation

```
f[s, t] = A + (B - A) s + (C - A) t
```

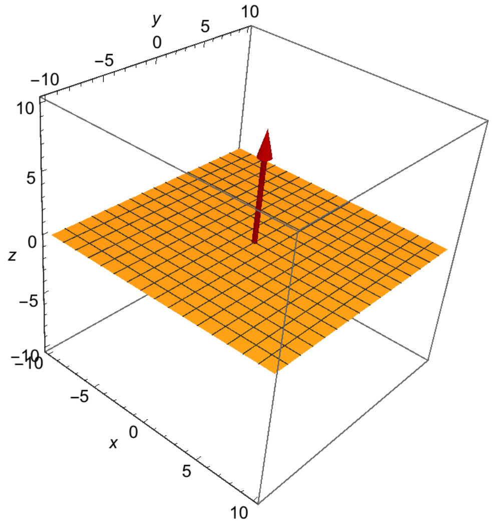
is the equation I want (with the proviso that A, B, and C are points which are not all co-linear). Three points on the xy-plane would be {1, 1, 0}, {-1, 1, 0}, and {1, -1, 0}.

```
aA = {1, 1, 0}
{1, 1, 0}

bB = {-1, 1, 0}
{-1, 1, 0}

cC = {1, -1, 0}
{1, -1, 0}

f[s_, t_] = aA + (bB - aA) s + (cC - aA) t
{1 - 2 s, 1 - 2 t, 0}
```

The above might be a good way to do it for a generalized plane, but the text has something different in mind for the xy-plane of this particular problem, like the following.

```
p1 = ParametricPlot3D[{u, v, 0}, {u, -10, 10}, {v, -10, 10},
   AxesLabel → {x, y, z}, ImageSize → 250, BoxRatios → {1, 1, 1}];

p2 = Graphics3D[
   {{Red, Arrowheads[0.06], Arrow[Tube[{{0, 0, 0}, {0, 0, 9}}, 0.2]]}}];
```

```
Show[p1, p2, PlotRange → {{-10, 10}, {-10, 10}, {-10, 10}}]
```



I found that I had to precalculate the normal vector and insert it in final form. And I could not work out a way to display the vector directly in **ParametricPlot3D**; it had to be done with **Show**.

3. Cone **r**(u, v) = {u Cos[v], u Sin[v], c u}

In[17]:=
```
Clear["Global`*"]
```

The formulation given in the problem description, in this case, is all I need for a plot. For example, with c= – 1.5, and starting with the function ‘con’, I get

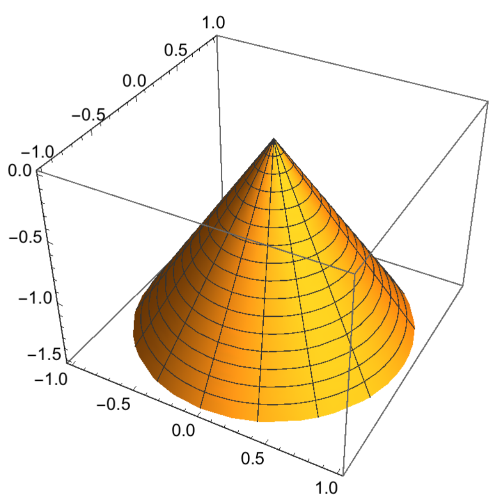
In[18]:=
```
con[u_, v_] = { u Cos[v], u Sin[v], -1.5 u}
```

Out[18]=
```
{u Cos[v], u Sin[v], -1.5 u}
```

The constant c being less than zero makes the tip go at the top vertically.

In[27]:=
```
p1 = ParametricPlot3D[con[u, v], {v, 0, 2 Pi}, {u, 0, 1}, ImageSize → 250]
```

Out[27]=



I can gather up a couple of vectors inhabiting the surface of the cone.

In[68]:=
```
c1 = con[0.8, 0.6]
```
Out[68]=
```
{0.660268, 0.451714, -1.2}
```

In[53]:=
```
c2 = con[0.5, -0.8]
```
Out[53]=
```
{0.348353, -0.358678, -0.75}
```

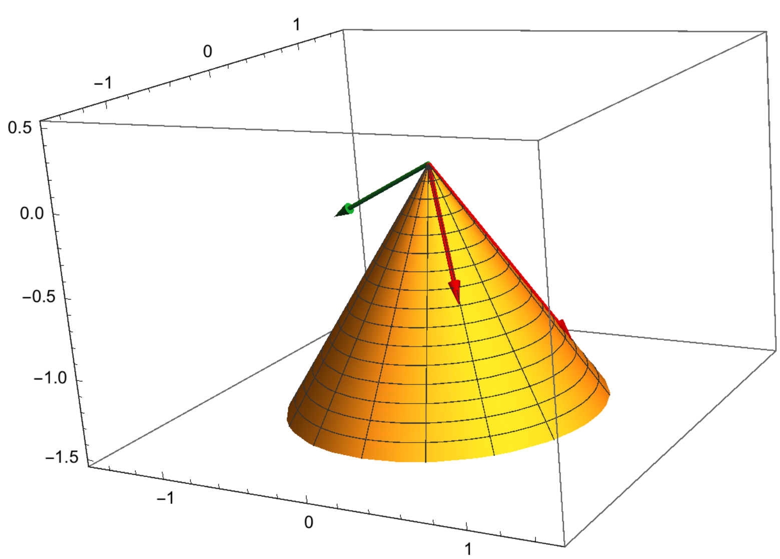
And crossing them, get a vector which is normal to the surface of the cone. Technically, it is normal to the surface at the tip.

In[71]:=
```
c3 = Cross[c1, c2]
```
Out[71]=
```
{-0.769199, 0.0771773, -0.39418}
```

In[74]:=
```
p2 = Graphics3D[
   {{Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0}, {0.6602684919277427`,
          0.4517139787160283`, -1.2000000000000002`}}, 0.014]]},
    {Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0},
         {0.3483533546735827`, -0.3586780454497614`, -0.75`}}, 0.014]]},
    {RGBColor[0.1, 0.7, 0.2], Arrowheads[0.03],
     Arrow[Tube[{{0, 0, 0}, {-0.7691991385767349`,
          0.07717734333750774`, -0.3941798919953841`}}, 0.014]]}}];
```

In[78]:=
```
Show[p1, p2,
 PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}, {-1.5, 0.5}}, ImageSize → 400]
```

Out[78]=



By taking partial derivatives for calculating the normal vector, I can get a more generalized one than the one shown above.

```
ru = D[{u Cos[v], u Sin[v], c u}, u]
{Cos[v], Sin[v], c}
```

```
rv = D[{u Cos[v], u Sin[v], c u}, v]
{-u Sin[v], u Cos[v], 0}
```

```
cprod = Cross[ru, rv]
```

$\{-c\,u\,\mathrm{Cos}[v], -c\,u\,\mathrm{Sin}[v], u\,\mathrm{Cos}[v]^2 + u\,\mathrm{Sin}[v]^2\}$

$\mathrm{TrigReduce}[u\,\mathrm{Cos}[v]^2 + u\,\mathrm{Sin}[v]^2]$

```
u
```

$\mathrm{cprodf} = \mathrm{cprod}\ /.\ u\,\mathrm{Cos}[v]^2 + u\,\mathrm{Sin}[v]^2 \rightarrow u$

```
{-c u Cos[v], -c u Sin[v], u}
```

The above line holds the normal vector the problem was looking for.

5. Paraboloid of revolution **r**(u, v) = {u Cos[v], u Sin[v], u^2}

```
In[96]:= Clear["Global`*"]
```

This time it is a parabola. First I make the function directed by the problem description, which works fine for the parametric plot.

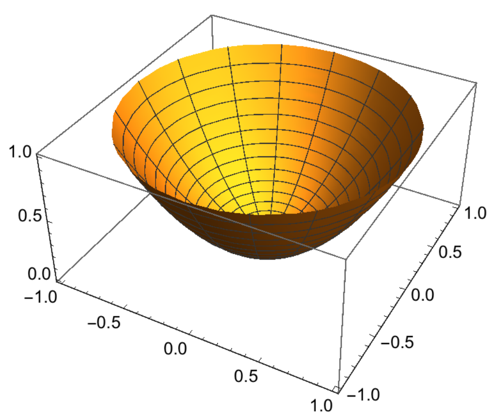
In[97]:= $\mathrm{para}[u\_, v\_] = \{u\,\mathrm{Cos}[v], u\,\mathrm{Sin}[v], u^2\}$

Out[97]= $\{u\,\mathrm{Cos}[v], u\,\mathrm{Sin}[v], u^2\}$

If I make u^2 negative, the hump will be on top. However, I will not do that, because I will need to see both sides anyway.

```
In[98]:= p1 = ParametricPlot3D[para[u, v], {v, 0, 2 Pi}, {u, 0, 1}, ImageSize → 250]
```

Out[98]=



I can find a couple of random vectors which contribute to the function surface.

```
In[119]:= pr1 = para[0.8, 0.1]
Out[119]= {0.796003, 0.0798667, 0.64}
```

In[124]:=
```
pr2 = para[0.8, -0.6]
```
Out[124]=
```
{0.660268, -0.451714, 0.64}
```

These vectors I can cross to get an example of a vector which is normal to the surface.

In[128]:=
```
pr3 = Cross[pr1, pr2]
```
Out[128]=
```
{0.340212, -0.0868703, -0.412299}
```

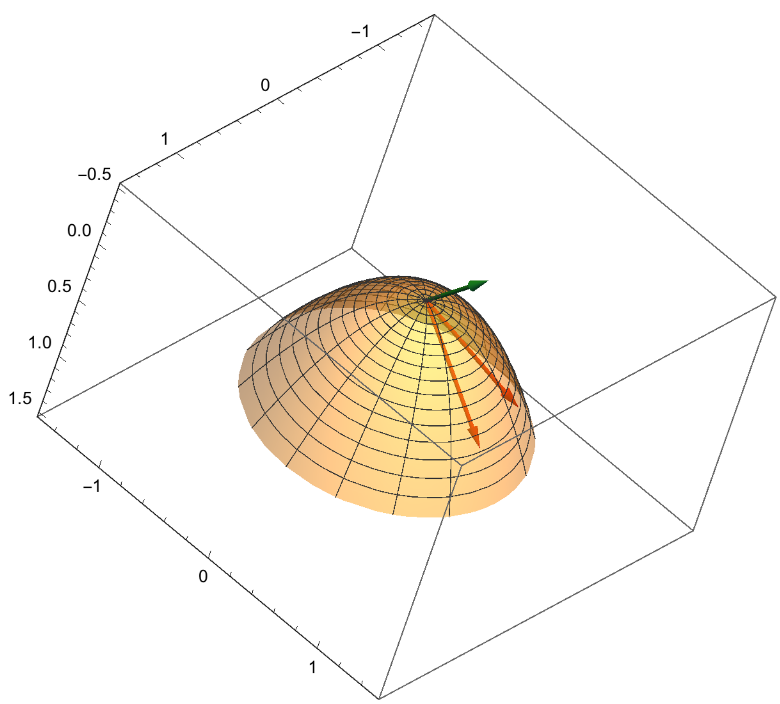
I rename an instance of the parametric plot in order to reduce its opacity.

In[137]:=
```
p2 = ParametricPlot3D[para[u, v], {v, 0, 2 Pi},
    {u, 0, 1}, ImageSize → 250, PlotStyle -> Opacity[0.5]];
```

In[133]:=
```
p3 = Graphics3D[
    {{Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0}, {0.7960033322224207`,
          0.07986673331746252`, 0.6400000000000001`}}, 0.014]]},
     {Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0}, {0.6602684919277427`,
          -0.4517139787160283`, 0.6400000000000001`}}, 0.014]]},
     {RGBColor[0.1, 0.7, 0.2], Arrowheads[0.03],
      Arrow[Tube[{{0, 0, 0}, {0.3402116557014342`,
          -0.08687029778859394`, -0.41229931983212237`}}, 0.014]]}}];
```

The green normal vector does not look particularly perpendicular to the function surface, but I think it meets the qualifications.

In[138]:=
```
Show[p2, p3, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}, {-.5, 1.5}},
 ImageSize → 400, ViewPoint → {800, 1000, 5000}]
```
Out[138]=



Using partial derivatives, I can create a more general expression for a normal vector.

```
ru = D[{u Cos[v], u Sin[v], u^2}, u]
{Cos[v], Sin[v], 2 u}

rv = D[{u Cos[v], u Sin[v], u^2}, v]
{-u Sin[v], u Cos[v], 0}

cprod = Cross[ru, rv]
{-2 u^2 Cos[v], -2 u^2 Sin[v], u Cos[v]^2 + u Sin[v]^2}

cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 → u
{-2 u^2 Cos[v], -2 u^2 Sin[v], u}
```

The above line contains the normal vector expression, which agrees with the text's answer.

7. Ellipsoid r(u, v) = {a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}

```
In[144]:= Clear["Global`*"]
```

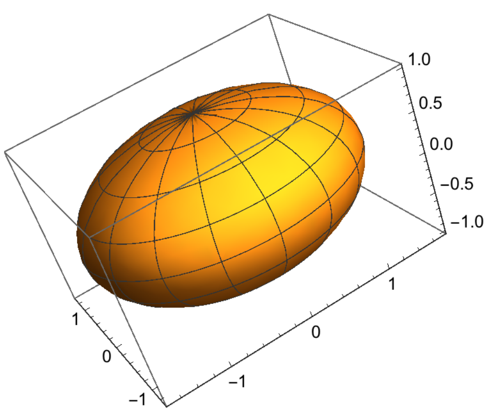
Thanks to *MathWorld* for a function form which applied the problem description in a useful way.

```
In[158]:= ellip[a_, b_, c_, u_, v_] = {a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}
Out[158]= {a Cos[u] Cos[v], b Cos[v] Sin[u], c Sin[v]}
```

I make the establishing plot.

```
In[159]:= p1 = ParametricPlot3D[ellip[1.7, 1.3, 1, u, v],
   {v, 0, 2 Pi}, {u, 0, 2 π}, ImageSize → 250]
```

Out[159]=



Add a couple of vectors which contribute to the surface.

```
In[160]:= pr1 = ellip[1.7, 1.3, 1, 0.8, 0.1]
Out[160]= {1.17848, 0.927904, 0.0998334}
```

In[169]:=
```
pr2 = ellip[1.7, 1.3, 1, 0.8, -0.6]
```
Out[169]=
```
{0.977529, 0.769677, -0.564642}
```

Cross these to get a vector normal to the surface.

In[179]:=
```
pr3 = 2.5 Cross[pr1, pr2]
```
Out[179]= $\{-1.50193, 1.90753, 3.00176\times10^{-16}\}$

Make another plot with reduced opacity to make the vectors easier to see.

In[184]:=
```
p2 = ParametricPlot3D[ellip[1.7, 1.3, 1, u, v], {v, 0, 2 Pi},
   {u, 0, 2 π}, ImageSize → 250, PlotStyle → Opacity[0.3]];
```

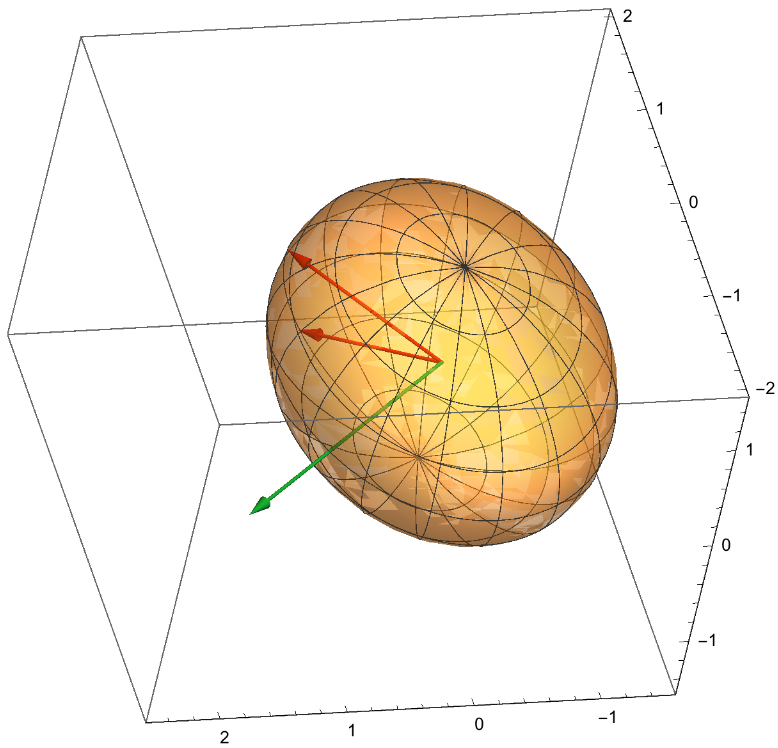
Make a separate graphic for the arrows, since I know of no way to tack them on to a parametric plot.

In[180]:=
```
p3 = Graphics3D[
   {{Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0}, {1.1784843322218799`,
          0.9279039879623635`, 0.09983341664682815`}}, 0.014]]},
    {Red, Arrowheads[0.03], Arrow[Tube[{{0, 0, 0}, {0.9775286626302601`,
          0.7696773895092891`, -0.5646424733950354`}}, 0.014]]},
    {RGBColor[0.1, 0.7, 0.2], Arrowheads[0.03],
     Arrow[Tube[{{0, 0, 0}, {-1.501933815866775`,
          1.9075308361591057`, 3.0017574092317646`*^-16}}, 0.014]]}}];
```

Show the current vector system, which includes a vector normal to the surface. It was necessary to extend the vector in order to penetrate the surface. In contrast to the last couple of problems, this normal vector does actually look perpendicular to the surface.

In[183]:=
```
Show[p2, p3, PlotRange → {{-2, 2}, {-1.5, 2.5}, {-1.5, 1.5}},
 ImageSize → 400, ViewPoint → {800, 1000, 5000}]
```

Out[183]=



In order to get a more general expression for a normal vector I stir up some partials.

```
ru = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, u]
{-a Cos[v] Sin[u], b Cos[u] Cos[v], 0}
```

```
rv = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, v]
{-a Cos[u] Sin[v], -b Sin[u] Sin[v], c Cos[v]}
```

```
cprod = Cross[ru, rv]
{b c Cos[u] Cos[v]^2, a c Cos[v]^2 Sin[u],
 a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]}
```

```
PossibleZeroQ[
 (a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]) - (a b Cos[v] Sin[v])]
True
```

The PZQ above verifies the equality.

```
cprodf = cprod /.
  a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v] -> a b Cos[v] Sin[v]
```

```
{b c Cos[u] Cos[v]^2, a c Cos[v]^2 Sin[u], a b Cos[v] Sin[v]}
```

The above line contains the text answer for the normal vector.

11. Satisfying numbered line (4), p. 441. Represent the paraboloid in problem 5 so that $\tilde{N}(0, 0) \neq 0$ and show $\tilde{N}$.

In[305]:=
```
Clear["Global`*"]
```

Pulling in some of the working material from problem 5. I'm not sure if it is the exact expectation from this problem, but I'm going to proceed to find the tangent plane to a given point using the method of *http://tutorial.math.lamar.edu/Classes/CalcIII/ParametricSurfaces.aspx.* If it's not the expected outcome, it is definitely related to it.

I see that I have, or can easily have, the function expressed in the

$\vec{r}(u, v) = \square\mathbf{i} + \square\mathbf{j} + \square\mathbf{k}$

standard format.

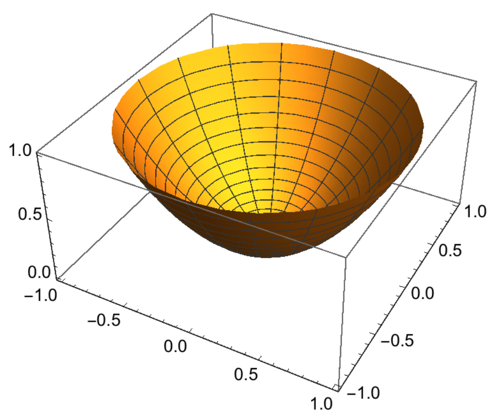
In[306]:=
```
para[u_, v_] = {u Cos[v], u Sin[v], u^2}
```

Out[306]= $\{u \operatorname{Cos}[v], u \operatorname{Sin}[v], u^2\}$

A reminder of the function's surface.

In[307]:=
```
p1 = ParametricPlot3D[para[u, v], {v, 0, 2 Pi}, {u, 0, 1}, ImageSize → 250]
```

Out[307]=



First step is to cross the partial derivatives.

In[308]:=
```
dupart = D[para[u, v], u]
```

Out[308]= {Cos[v], Sin[v], 2 u}

In[309]:=
```
dvpart = D[para[u, v], v]
```

Out[309]= {-u Sin[v], u Cos[v], 0}

In[310]:=
```
cdv = Cross[dupart, dvpart]
```

Out[310]= $\{-2 u^2 \operatorname{Cos}[v], -2 u^2 \operatorname{Sin}[v], u \operatorname{Cos}[v]^2 + u \operatorname{Sin}[v]^2\}$

After crossing the partials, I need to get the assigned value for u and v which produced the point of tangency. This can be calculated if not already known, but in this case it is known.

It is the pr1 point below, and was generated with u=0.8 and v=0.1.

In[311]:=
```
p2 = Graphics3D[{{Blue, PointSize[0.01], Point[{0.7960033322224207`,
      0.07986673331746252`, 0.6400000000000001`}]}}];
```

In[312]:=
```
pr1 = para[0.8, 0.1]
```

Out[312]=
```
{0.796003, 0.0798667, 0.64}
```

Adapting the general normal vector to this point gives me

In[239]:=
```
cdvs = cdv /. {u → 0.8, v → 0.1}
```

Out[239]=
```
{-1.27361, -0.127787, 0.8}
```

And the tangent plane can be written immediately as

In[259]:=
```
tangentplane =
  -1.2736053315558733` (x - 0.7960033322224207`) - 0.12778677330794005`
     (y - 0.07986673331746252`) + 0.8` (z - 0.6400000000000001`) == 0
```

Out[259]=
```
-1.27361 (-0.796003 + x) - 0.127787 (-0.0798667 + y) + 0.8 (-0.64 + z) == 0
```

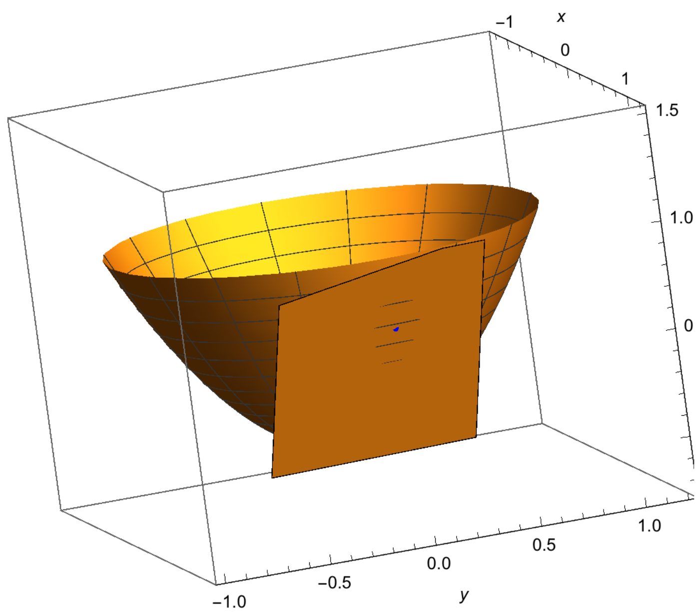
which I can plug into an equation tolerant plot type like **ContourPlot3D**.

In[303]:=
```
p4 = ContourPlot3D[{-1.2736053315558733` (-0.7960033322224207` + x) -
      0.12778677330794005` (-0.07986673331746252` + y) +
      0.8` (-0.6400000000000001` + z) == 0},
   {x, -0.5`, 1`}, {y, -0.5`, 0.5`}, {z, 0, 1}, Mesh → None];
```

Then all three plots, each of a different kind, can be shown together. The blue point of tangency can be clearly seen.

In[304]:=
```
Show[p1, p2, p4, PlotRange → {{-1.25, 1.25}, {-1, 1.2}, {-0.25, 1.5}},
 ImageSize → 400, (*ViewPoint→{80,100,500},*)
 AxesLabel → {x, y, z}, ImagePadding → {{20, 20}, {10, 70}}]
```

Out[304]=



No green cell on this problem, because the text answer apparently has a specific point for tangency in mind, which I can't decipher.

13. Representation z = f (x, y). Show that z = f (x, y) or g = z − f (x, y) = 0 can be written $\left(f_u = \frac{\partial f}{\partial u},\ \text{etc.}\right)$

Even with treatment in the s.m., I don't understand what is supposed to happen with the present problem. In the s.m. an example plane is processed, and some of the same steps shown above in problem 11 are carried out, viz calculation of partial derivatives and their Cross product. There is a dangling numbered line (6) , p. 443, which may or may not be part of this problem, but it is adjacent to it, and the s.m. thinks it is significant.

14 - 19 Derive a parametric representation

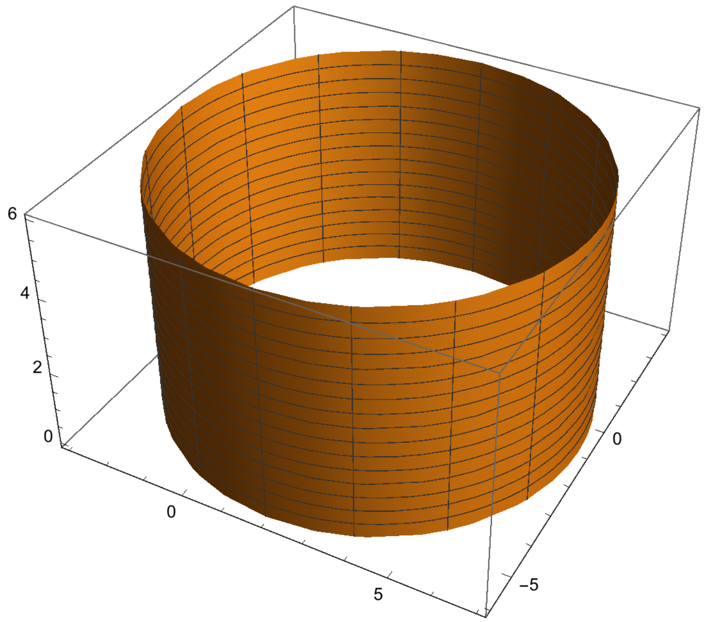
Find a normal vector. The answer gives one representation; there are many. Sketch the surface and parameter curves.

15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$

```
Clear["Global`*"]
```

The constants just specify the location of the center of the cylinder.

```
ParametricPlot3D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u, 0, 2 π}, {v, 0, 6}]
```



The parametric curves are shown in the figure. In the constant-u plane they are circles. In the constant-v plane they are straight lines.

```
par[u_, v_] = {2 + 5 Cos[u], -1 + 5 Sin[u], v}
{2 + 5 Cos[u], -1 + 5 Sin[u], v}
```

Above is the parametric equation for the cylinder. Below I will take the partial derivatives so I can find a normal vector by the cross product of them.

```
fir = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u}]
{-5 Sin[u], 5 Cos[u], 0}
```

```
sec = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {v}]
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{5 Cos[u], 5 Sin[u], 0}
```

The vector shown in the line above is a normal vector, and agrees with the text's answer.

17. Sphere $x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 2.25$

```
Clear["Global`*"]
```

Again, the constants 2.8 and -3.2 merely locate the center of the sphere. In the cartesian formula for the sphere, the quantity 2.25 above is the square of the radius.

```
radd = (2.25)^.5
```

```
1.5
```

And so my parametric version will look like

```
parsph[u_, v_] =
 {1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}
```

```
{1.5` Cos[v] Sin[u], -2.8` + 1.5` Sin[u] Sin[v], 3.2` + 1.5` Cos[u]}
```

and its plot

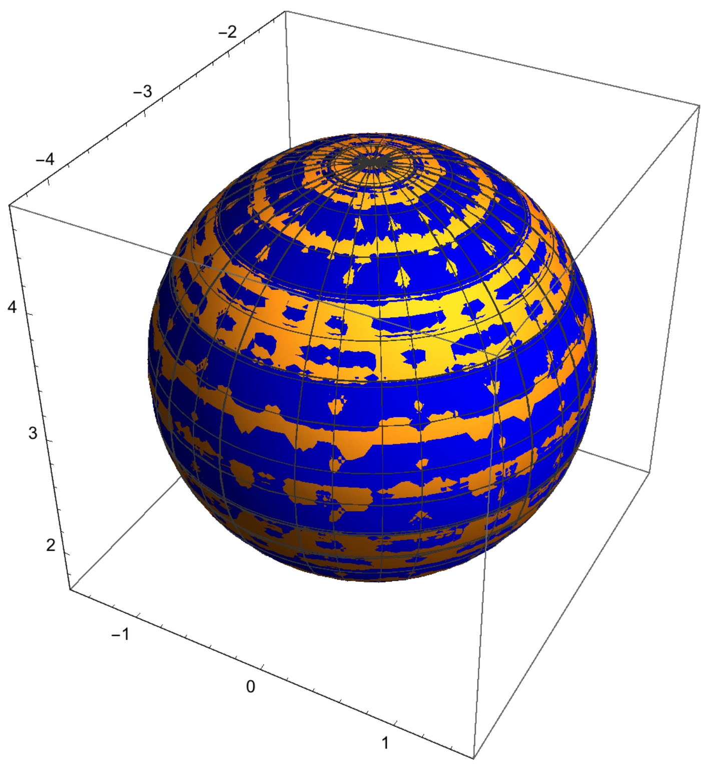
```
In[328]:= p1 = ParametricPlot3D[{1.5 Sin[u] Cos[v],
     1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u, -7, 7}, {v, -7, 7}];
```

whereas that of the text answer looks like

```
In[327]:= p2 = ParametricPlot3D[{1.5 Cos[u] Cos[v], -2.8 + 1.5 Sin[u] Cos[v],
     3.2 + 1.5 Sin[v]}, {u, -7, 7}, {v, -7, 7}, PlotStyle → Blue];
```

```
In[326]:= Show[p1, p2]
```

Out[326]=



The Christmas tree ball above shows that the answer I created is the same as the text's. However, comparison of the actual plot expressions shows that the nuts and bolts are quite different. Hence the general normal in the cell below is colored yellow and not green.

```
fir = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u}]
```

```
{1.5 Cos[u] Cos[v], 1.5 Cos[u] Sin[v], -1.5 Sin[u]}
```

```
sec = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {v}]
{-1.5 Sin[u] Sin[v], 1.5 Cos[v] Sin[u], 0}
```

```
norm = Simplify[Cross[fir, sec]]
```

$\{2.25 \, \mathrm{Cos}[v] \, \mathrm{Sin}[u]^2, \, 2.25 \, \mathrm{Sin}[u]^2 \, \mathrm{Sin}[v], \, 2.25 \, \mathrm{Cos}[u] \, \mathrm{Sin}[u]\}$

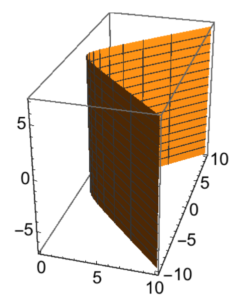
19. Hyperbolic cylinder $x^2 - y^2 = 1$

```
Clear["Global`*"]
```

I found a reference to a hyperbolic identity which is analogous to $\sin^2 x + \cos^2 x = 1$. It is $\cosh^2 x - \sinh^2 x = 1$. Since the squared exponent is already in the problem expression, there is no need to repeat it.

```
parhyp[u_, v_] = {Cosh[u], Sinh[u], v}
{Cosh[u], Sinh[u], v}
```

```
ParametricPlot3D[{Cosh[u], Sinh[u], v}, {u, -3, 3}, {v, -7, 7}]
```



As for parameter curves, I think the ones in constant-u planes are hyperbolas (or at least half hyperbolas). The ones in constant-v planes are straight lines.

```
fir = D[{Cosh[u], Sinh[u], v}, {u}]
{Sinh[u], Cosh[u], 0}
```

```
sec = D[{Cosh[u], Sinh[u], v}, {v}]
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{Cosh[u], -Sinh[u], 0}
```

The above line agrees with the normal vector contained in the text’s answer.